Clase 5: Propiedades de la derivada

C. J. Vanegas

27 de abril de 2008

1. Propiedades de la derivada.

i Sea $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ diferenciable en $\overline{x}_0$ y sea $c \in \mathbb{R}$. Entonces. $h(\overline{x}) = cf(\overline{x})$ es diferenciable en $\overline{x}_0$ y $Dh(\overline{x}_0) = cDf(\overline{x}_0)$.

Demostración :

$$\lim_{\overline{x}\to\overline{x}_0} \frac{||cf(\overline{x}) - cf(\overline{x}_0) - cDf(\overline{x}_0)(\overline{x}-\overline{x}_0)||}{||\overline{x}-\overline{x}_0||} = 0.$$

□

ii Suma: Sean$f, g : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ diferenciables en $\overline{x}_0$. Entonces. $h(\overline{x}) = f(\overline{x}) + g(\overline{x})$ es diferenciable en $\overline{x}_0$ y $Dh(\overline{x}_0) = Df(\overline{x}_0) + Dg(\overline{x}_0)$.

Demostración :

$$\lim_{\overline{x}\to\overline{x}_0} \frac{||f(\overline{x}) + g(\overline{x}) - (f(\overline{x}_0) + g(\overline{x}_0)) - (Df(\overline{x}_0) + Dg(\overline{x}_0))(\overline{x}-\overline{x}_0)||}{||\overline{x}-\overline{x}_0||} \leq$$
$$\lim_{\overline{x}\to\overline{x}_0} \frac{f(\overline{x}) - f(\overline{x}_0) - Df(\overline{x}_0)(\overline{x}-\overline{x}_0)}{||\overline{x}-\overline{x}_0||} + \lim_{\overline{x}\to\overline{x}_0} \frac{g(\overline{x}) - g(\overline{x}_0) - Dg(\overline{x}_0)(\overline{x}-\overline{x}_0)}{||\overline{x}-\overline{x}_0||} = 0.$$

□

iii Producto: Sean$f, g : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ diferenciables en $\overline{x}_0$. Entonces. $h(\overline{x}) = f(\overline{x})g(\overline{x})$ es diferenciable en $\overline{x}_0$ y $Dh(\overline{x}_0) = g(\overline{x}_0)Df(\overline{x}_0) + f(\overline{x}_0)Dg(\overline{x}_0)$.

Demostración :

$$\lim_{\overline{x}\to\overline{x}_0} \frac{||f(\overline{x})g(\overline{x}) - (f(\overline{x}_0)g(\overline{x}_0)) - (g(\overline{x}_0)Df(\overline{x}_0) + f(\overline{x}_0)Dg(\overline{x}_0))(\overline{x}-\overline{x}_0)||}{||\overline{x}-\overline{x}_0||} \leq$$

$$\lim_{\overline{x}\to\overline{x}_0} |g(\overline{x}_0)| \frac{f(\overline{x}) - f(\overline{x}_0) - Df(\overline{x}_0)(\overline{x}-\overline{x}_0)}{||\overline{x}-\overline{x}_0||} + \lim_{\overline{x}\to\overline{x}_0} |f(\overline{x}_0)| \frac{g(\overline{x}) - g(\overline{x}_0) - Dg(\overline{x}_0)(\overline{x}-\overline{x}_0)}{||\overline{x}-\overline{x}_0||}$$

$$+ \lim_{\overline{x}\to\overline{x}_0} \frac{||f(\overline{x}) - f(\overline{x}_0)|| \, ||g(\overline{x}) - g(\overline{x}_0)||}{||\overline{x}-\overline{x}_0||} = 0$$

□

iv <u>Cociente:</u> Sean$f, g : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ diferenciables en $\overline{x}_0$ y $g(U) \leq 0$. Entonces. $h(\overline{x}) = \dfrac{f(\overline{x})}{g(\overline{x})}$ es diferenciable en $\overline{x}_0$ y $Dh(\overline{x}_0) = \dfrac{g(\overline{x}_0)Df(\overline{x}_0) - f(\overline{x}_0)Dg(\overline{x}_0)}{(g(\overline{x}_0))^2}$.

Demostración : Tomamos $h(\overline{x})$ como $\dfrac{1}{g(\overline{x})}f(\overline{x})$ y aplicamos el resultado anterior. □

Ejemplo 1. 1 ☞ *Sea* $f, g : \mathbb{R}^3 \longrightarrow \mathbb{R}$. *Donde* $f(\overline{x}) = x^2 + y^2 + z^2$ *y*$g(\overline{x}) = z^2 + 2$ *encontrar la derivada de* $h(x, y, z) = f(\overline{x})g(\overline{x})$ *en* $\overline{x} = (x, y, z) \in \mathbb{R}^3$.
Solución:

$$Df(\overline{x}) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix}_{1\times 3} = (2x \; 2y \; 2z)$$

$$Dg(\overline{x}) = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix}_{1\times 3} = (0 \quad 0 \quad 2z)$$

Luego utilizando la propiedad del producto, obtenemos:

$$Dh(\overline{x}) = g(\overline{x})Df(\overline{x}) + f(\overline{x})Dg(\overline{x})$$

$$= (x^2 + y^2 + z^2)(0 \quad 0 \quad 2z) + (z^2 + 2)(2x \quad 2y \quad 2z)$$

$$= ((z^2 + 2)2x \quad (z^2 + 2)2y \quad (x^2 + y^2 + 2z^2 + 2)2z)$$

$$= ((z^2 + 2)2x \quad (z^2 + 2)2y \quad 4z^3 + 4z + (x^2 + y^2)).$$

2 ☞ *Sea* $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$. *Donde* $f(x, y) = x + y$ *y*$g(x, y) = x^2 + y^2 + 1$ *encontrar la derivada de* $h(x, y, z) = \dfrac{f(\overline{x})}{g(\overline{x})}$ *en* $\overline{x} = (x, y) \in \mathbb{R}^2$.
Solución:

$$Df(\overline{x}) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}_{1\times 2} = (1 \quad 1)$$

$$Dg(\overline{x}) = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{1\times 2} = (2x \; 2y)$$

con $g(\overline{x}) \leq 0 \quad \forall \overline{x} \in \mathbb{R}^3$. *Luego utilizando la propiedad del cociente, obtenemos:*

$$\begin{aligned}
Dh(\overline{x}) &= \frac{g(\overline{x})Df(\overline{x}) - f(\overline{x})Dg(\overline{x})}{(g(\overline{x}))^2} \\
&= \frac{(x^2+y^2+1)(1 \quad 1) - (x+y)(2x \quad 2y)}{(x^2+y^2+1)} \\
&= \begin{pmatrix} \frac{x^2+y^2+1-(x+y)2x}{(x^2+y^2+1)^2} & \frac{x^2+y^2+1-(x+y)2y}{(x^2+y^2+1)^2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{-x^2+y^2+1-2xy}{(x^2+y^2+1)^2} & \frac{x^2-y^2+1-2xy}{(x^2+y^2+1)^2} \end{pmatrix}
\end{aligned}$$

2. Regla de derivación para funciones compuestas

Teorema 1. *Regla de la cadena.*
Supongamos que:

1. $U, V \subset \mathbb{R}^n$ *son conjuntos abiertos.*
2. $g : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *y* $f : V \subset \mathbb{R}^m \longrightarrow \mathbb{R}^p$ *tales que* $g(U) \subset V$.
3. g *es diferenciable en* $\overline{x}_0$.
4. f *es diferenciable en* $g(\overline{x}_0)$. *Entonces:*
   $f \circ g$ *es diferenciable en* $\overline{x}_0$ *y* $D(f \circ g)(\overline{x}_0) = Df(g(\overline{x}_0)) \cdot Dg(\overline{x}_0)$. *Donde* $D(f \circ g)(\overline{x}_0)$ *es una matriz* $p \times n$, $Df(g(\overline{x}_0))$ *es una matriz* $p \times m$ *y* $Dg(\overline{x}_0)$ *es una matriz* $m \times n$.

Ver el libro de J. Marsden y A. Tromba. Cálculo Vectorial.

Observación 1. *Si* $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ *y* $c : \mathbb{R} \longrightarrow \mathbb{R}^3$ *donde* $c(t) = (x(t), y(t), z(t)) = (c_1, c_2, c_3)$ *entonces:*
$f(c(t)) = f(x(t), y(t), z(t))$ *y* $D(f \circ c)(t) = D(f(c(t)))D(c(t))$
Note que:

☛ $D(f(\overline{x})) = Df(x, y, z) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix}_{1\times 3}$

☛ $D(c(t)) = \begin{pmatrix} \frac{\partial c_1}{\partial t} \\ \frac{\partial c_2}{\partial t} \\ \frac{\partial c_3}{\partial t} \end{pmatrix}_{3\times 1} = \begin{pmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial t} \end{pmatrix}$

☛ $Df(c(t)) = \begin{pmatrix} \frac{\partial f(c(t))}{\partial x} & \frac{\partial f(c(t))}{\partial y} & \frac{\partial f(c(t))}{\partial z} \end{pmatrix}$

Por otro lado:

$$D(f\circ c)(t) = D(f(c(t)))D(c(t)) = \begin{pmatrix} \frac{\partial f(c(t))}{\partial x} & \frac{\partial f(c(t))}{\partial y} & \frac{\partial f(c(t))}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial c_1}{\partial t} \\ \frac{\partial c_2}{\partial t} \\ \frac{\partial c_3}{\partial t} \end{pmatrix}$$

$$= \frac{\partial f(c(t))}{\partial x}\frac{dx}{dt} + \frac{\partial f(c(t))}{\partial y}\frac{dy}{dt} + \frac{\partial f(c(t))}{\partial z}\frac{dz}{dt}$$

$$= \nabla f(c(t))\cdot c'(t)$$

donde $c'(t) = (x'(t), y'(t), z'(t))$ *En Conclusión:*

Si $\boxed{h(t) = f(c(t)) = f(x(t), y(t), z(t))}$. *Entonces.* $\boxed{\frac{\partial h}{\partial t} = \frac{\partial f}{\partial x}()\frac{dx}{dt} + \frac{\partial f}{\partial y}()\frac{dy}{dt} + \frac{\partial f}{\partial z}()\frac{dz}{dt}}$

Observación 2. *Sean* $f:\mathbb{R}^3 \longrightarrow \mathbb{R}$ *y* $g:\mathbb{R}^3 \longrightarrow \mathbb{R}$ *donde* $g(x,y,z) = g(\overline{x}) = (u(\overline{x}), v(\overline{x}), w(\overline{x}))$

Calculemos:

$D(f\circ g)(\overline{x}) = D(f(g(\overline{x})))D(g(\overline{x}))$

Note que:

☛ $f(\overline{x}) = f(x,y,z)$

☛ $D(f(\overline{x})) = Df(x,y,z) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix}_{1\times 3}$

☛ $(f\circ g)(\overline{x}) = f(u(\overline{x}), v(\overline{x}), w(\overline{x}))$

☛ $Df(g(\overline{x})) = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \end{pmatrix}$

☛ $D(g(\overline{x})) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$ *Así podemos escribir:*

$$Dh(\overline{x}) = D(f\circ g)(\overline{x}) = D(f(g(\overline{x})))D(g(\overline{x}))$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$$

$$= \begin{pmatrix} \underbrace{\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}}_{\frac{\partial h}{\partial x}} & \underbrace{\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}}_{\frac{\partial h}{\partial y}} & \underbrace{\frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z}}_{\frac{\partial h}{\partial z}} \end{pmatrix}$$

3. Interpretación geométrica de la regla de la cadena.

Sea $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $c : \mathbb{R} \longrightarrow \mathbb{R}^2$.

✌ $c(t) = (x(t), y(t))$ Se piensa como la trayectoria en el plano. (vector de posición)

✌ $c'(t) = (x'(t), y'(t))$ Se piensa como el vector tangente e la trayectoria en $c(t)$. (vector velocidad)

Donde $p(t) = (f \circ c)(t) = f(c(t))$ y $p'(t) = Df(c(t)) \cdot c'(t)$

Ejemplo 2. 1 ☞ *Calcule $Dh(x, y)$, si $h(x, y) = f(u(x, y), v(x, y))$;*

$f(u, v) = \dfrac{u^2 + v^2}{u^2 - v^2}$, $u(x, y) = e^{-x-y}$ $yv(x, y) = e^{xy}$

Solución:

Consideremos: ☛ $\overline{x} = (x, y)$

☛ $g(x, y) = (u(x, y), v(x, y))$

☛ $h(x, y) = (f \circ g)(x, y) = f(g(x, y)) = f(u(\overline{x}), v(\overline{x}))$ *Por otro lado:*

$$Dh(x, y) = Df(g(x, y)) \cdot Dg(x, y) = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

$$\left(\underbrace{\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}}_{\frac{\partial h}{\partial x}} \quad \underbrace{\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}}_{\frac{\partial h}{\partial y}} \right)$$

$$\frac{\partial h}{\partial x} = \frac{(u^2 - v^2)2u - (u^2 + v^2)2u}{(u^2 - v^2)^2}(-1)e^{-x-y} + \frac{(u^2 - v^2)2v - (u^2 + v^2)(-2v)}{(u^2 - v^2)^2}$$
$$- \frac{4uv^2e^{-x-y}}{(u^2 - v^2)^2} + \frac{4u^2vye^{xy}}{(u^2 - v^2)^2}$$

$$\frac{\partial h}{\partial y} = -\frac{4uv^2e^{-x-y}}{(u^2 - v^2)^2} + \frac{4u^2vxe^{xy}}{(u^2 - v^2)^2}$$

✎ *Decimos que una función es c^1 si existen sus derivadas parciales y estas son continuas.*

2 ☞ *El gradiente es normal a las superficies de nivel:*

Si $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ en c^1 y $(x_0, y_0, z_0) = (\overline{x}_0)$ es un punto sobre la superficie de nivel $s := f(x, y, z) = k$ tal que $\nabla f(x_0, y_0, z_0)$ es normal a la superficie de nivel: es decir si V es el vector tangente en t_0 de una trayectoria $c(t) = (x(t), y(t), z(t))$ en S con

$c(t_0) = (x_0, y_0, z_0) = (x(t_0), y(t_0), z(t_0))$ *entonces* $\nabla f \cdot V = 0$ *Como* $c(t)$ *esta en* S, *entonces* $f(c(t)) = k$ *luego* $\underbrace{\frac{d}{dt} f(c(t))\Big|_{t=t_0} = 0}_{\frac{\partial f}{\partial x}()\frac{\partial x}{\partial t}()+\frac{\partial f}{\partial y}()\frac{\partial y}{\partial t}()+\frac{\partial f}{\partial z}()\frac{\partial z}{\partial t}()=0}$

$\Rightarrow \nabla f(c(t_0)) \cdot c' t_0 = 0$

☞ EJERCICIOS

✍ $h(x,y) = f(x, u(x,y))$, donde $u(x,y) = x^2y$. Vamos a calcular $\frac{\partial h}{\partial x}$ y $\frac{\partial h}{\partial y}$

Solución:

$h(x,y) = f(x, u(x,y)) = f(v(x), u(x,y))$, donde $v(x) = x$ y $u(x,y) = x^2y$, así:

$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial f}{\partial v} + \frac{\partial f}{\partial u} 2xy$

$\frac{\partial h}{\partial y} = \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} = \frac{\partial f}{\partial u} x^2$

✍ Sean $u : \mathbb{R}^3 \Rightarrow \mathbb{R}^2$, $f : \mathbb{R}^2 \Rightarrow \mathbb{R}^2$ y $S = (f \circ u) : \mathbb{R}^3 \Rightarrow \mathbb{R}^2$, donde: $S(x,y,z) = (f_1(u(\overline{x})), f_2(u(\overline{x}))) = (f_1(u_1(\overline{x}), u_2(\overline{x})), f_2(u_1(\overline{x}), u_2(\overline{x})))$

☛ $u_1(\overline{x}) = x^2yz^2$

☛ $u_2(\overline{x}) = y^2 + e^z + x$

$\frac{\partial S_2}{\partial z} = \frac{\partial f_2}{\partial u_1}\frac{\partial u_1}{\partial z} + \frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial z} = \frac{\partial f_2}{\partial u_1} 2x^2yz + \frac{\partial f_2}{\partial u_2} e^z.$

$$DS(\overline{x}) = \begin{pmatrix} \frac{\partial s_1}{\partial x} & \frac{\partial s_1}{\partial y} & \frac{\partial s_1}{\partial z} \\ \frac{\partial s_2}{\partial x} & \frac{\partial s_2}{\partial y} & \frac{\partial s_2}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{pmatrix} \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \end{pmatrix}$$